Journal of Pure and Applied Algebra 150 (2000) 95–106

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

The representations of the coordinate ring of the quantum symplectic space

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Received 20 June 1998; received in revised form 10 November 1998

Communicated by C. Kassel

Abstract

In this paper, the irreducible representations of the coordinate ring $\mathcal{O}_q(sp\mathbb{C}^{2n})$ of quantum symplectic space are classified. Especially, the De Concini–Kac–Procesi conjecture is proved to be true for this algebra. The main steps are the computation of the center and the degree of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ by using the associated quasipolynomial algebra $\overline{\mathcal{O}_q(sp\mathbb{C}^{2n})}$, find some "good" points in the underlying symplectic leaves and construct some quasipolynomial algebras correspond to these "good" points.

1. Introduction

The q-analogues $S_q^+(V)$, $S_q^-(V)$ of the symmetric algebras, as Manin [3] did to reformulate the quantum algebra $\mathcal{O}_q[M_n(\mathbb{C})]$, were constructed by Takeuchi [6] and independently by Faddeev et al. [2]. Smith [5] called the algebra $S_q^+(V)$ the coordinate ring of quantum Euclidean space, and the algebra $S_q^-(V)$ the coordinate ring of quantum symplectic space, denoted by $\mathcal{O}_q(o\mathbb{C}^n)$ and $\mathcal{O}_q(sp\mathbb{C}^{2n})$, respectively. In [4], all of the primitive ideals of $\mathcal{O}_q(sp\mathbb{C}^{2n})$ are classified by Oh by using the so-called admissible sets when q is generic.

In quantum mechanics, the algebra of observables is noncommutative, and the objects which play the role of points are the irreducible representations of the algebra of

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0022-4049/00/$ - see front matter
PII: S0022-4049(99)00023-7

observables. Hence, it is natural to understand the irreducible representations of a quantum algebra. In this paper, the irreducible representations of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ are classified when q is a mth root of unity, for m odd, by proving that the De Concini–Kac–Procesi conjecture (DKP for short) for the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$.

The arrangement of the paper is as follows: In Section 2 the coordinate ring $\mathcal{O}_q(sp\mathbb{C}^{2n})$ of the quantum symplectic space is introduced. In Section 3 we review the theory developed by De Concini and Procesi about the degree of a prime algebra. In Section 4 the degree and the center of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ are computed by using the theory in Section 3. Finally, in Section 5 the irreducible representations of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ are classified.

The author would like to thank the referee for valuable comments.

2. Definitions and basic properties

In this section, we give the definition of the coordinate ring of the quantum symplectic space $\mathcal{O}_q(sp\mathbb{C}^{2n})$, defined in [2,6] see also [4].

Definition 2.1. The coordinate ring of the quantum symplectic space $\mathcal{O}_q(sp\mathbb{C}^{2n})$ is the algebra generated by $x_1,\ldots,x_{2n}$ satisfying the following relations:

$$\begin{aligned} &x_jx_i = qx_ix_j, \quad i<j,\ i'\neq j,\\ &x_{i'}x_i = q^2x_ix_{i'} + (q^2-1)\sum_{1\le k<i} q^{i-k}x_kx_{k'}, \quad i<i', \end{aligned} \tag{2.1}$$

where $i' = 2n-i+1$ for $1\le i\le 2n$.

Definition 2.2. An element $x\in\mathcal{O}_q(sp\mathbb{C}^{2n})$, is called covariant if for any x_i there exists an integer n_i such that

$$xx_i = q^{n_i}x_ix. \tag{2.2}$$

Clearly, x_n and x_{n+1} are covariant.

The following lemmas are taken from [4].

Lemma 2.3. *Put* $\Omega_i = \sum_{1\le k\le i} q^{i-k}x_kx_{k'},\ i=1,2,\ldots,n.$

(1) *The element* Ω_i *is covariant, more precisely,*

$$\begin{aligned} &\Omega_ix_k = q^2x_k\Omega_i, \quad 1\le k\le i;\\ &\Omega_ix_k = q^{-2}x_k\Omega_i, \quad i'\le k\le 2n;\\ &\Omega_ix_k = x_k\Omega_i, \quad i<k<i';\\ &\Omega_i\Omega_j = \Omega_j\Omega_i, \quad 1\le k\le n. \end{aligned} \tag{2.3}$$

(2) *We have the following rules*:

$$\begin{aligned} \Omega_i &= \sum_{j+1\le k\le i} q^{i-k} x_k x_{k'} + q^{i-j}\Omega_j, \\ x_{i'}x_i - q^2 x_i x_{i'} &= (q^2-1)q\Omega_{i-1}, \\ x_{i'}x_i - x_i x_{i'} &= (q^2-1)\Omega_i. \end{aligned} \tag{2.4}$$

Lemma 2.4. *The algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ is an iterated Ore extension. Thus it is a noetherian and integral domian.*

3. The degree of a prime algebra

The main tool used to compute the degree of $\mathcal{O}_q(sp\mathbb{C}^{2n})$ is the theory developed in [1] by De Concini and Procesi. So we shall first recall the set-up from there.

Let A be a prime algebra (i.e. $aAb = 0$ implies $a = 0$ or $b = 0$) over the complex numbers $\mathbb{C}$ and Z be the center of A. Then Z is a domain and A is a torsion free module over Z. Assume that A is a finite module over Z. Then A embeds in a finite dimensional central simple algebra $Q(A) = Q(Z) \otimes_Z A$, where $Q(Z)$ is the fraction field of Z. If $\overline{Q(Z)}$ denotes the algebraic closure of $Q(Z)$, we have that $\overline{Q(Z)} \otimes_Z A$ is the full algebra $M_d(\overline{Q(Z)})$ of $d \times d$ matrices over $\overline{Q(Z)}$. Then d is called the degree of A.

Let A be a prime algebra over $\mathbb{C}$ generated by $x_1, x_2, \dots, x_n$ and let Z_0 be a central subalgebra of A. For each $i = 1, 2, \dots, n$, denote by A^i the subalgebra of A generated by $x_1, x_2, \dots, x_i$ and let $Z_0^i = Z_0 \cap A^i$. We assume that the following conditions hold for each $i = 1, 2, \dots, n$:

1. $x_i x_j = b_{i,j} x_j x_i + p_{i,j}$ if $i > j$, where $b_{i,j} \in \mathbb{C}$ and $p_{i,j} \in A^{i-1}$.
2. A^i is a finite module over Z_0^i.
3. The formulas $\sigma_i(x_j) = b_{i,j} x_j$ for $j < i$ define an automorphism of A^{i-1} which is the identity on Z_0^{i-1}.
4. The formulas $D_i(x_j) = p_{i,j}$ for $j < i$ define a twisted derivation relative to σ_i.

It is easy to see that A then is an iterated twisted Ore extension. Let $\overline{A}$ be an associative algebra generated by $y_1, y_2, \dots, y_n$ with defining relations $y_i y_j = b_{i,j} y_j y_i$ for $j < i$. We call this algebra the associated quasipolynomial algebra of A. In [1] the following was proved:

Theorem 3.1. *Under the above assumptions the degree of A is equal to the degree of the associated quasipolynomial algebra $\overline{A}$.*

Given an $n \times n$ skew-symmetric matrix $H = (h_{i,j})$ over $\mathbb{Z}$ we construct the twisted polynomial algebra $\mathbb{C}_H[x_1, x_2, \dots, x_n]$ as follows: It is the algebra generated by elements $x_1, x_2, \dots, x_n$ with the following defining relations:

$$x_i x_j = q^{h_{i,j}} x_j x_i \quad \text{for } i, j = 1, 2, \dots, n. \tag{3.1}$$

The matrix H is called the defining matrix of the algebra $\mathbb{C}_H[x_1,x_2,\dots,x_n]$. It can be viewed as an iterated twisted polynomial algebra with respect to any ordering of the indeterminates x_i. Given $a=(a_1,a_2,\dots,a_n)\in\mathbb{Z}_+^n$ we write $x^a=x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$.

Let q be a primitive mth root of unity. We consider the matrix H as a matrix of the homomorphism $H:\mathbb{Z}^n\to(\mathbb{Z}/m\mathbb{Z})^n$ and we denote by K the kernel of H and by h the cardinality of the image of H. The following was proved in [1]:

Theorem 3.2. (a) *The monomials x^a with $a\in K\cap\mathbb{Z}_+^n$ form a basis of the center of $\mathbb{C}_H[x_1,x_2,\dots,x_n]$.*
(b) *degree $\mathbb{C}_H[x_1,x_2,\dots,x_n]=\sqrt{h}$.*

It is well known that a skew-symmetric $n\times n$ matrix over $\mathbb{Z}$ such as our matrix H can be brought into a block diagonal form by an element $W\in SL_n(\mathbb{Z})$. Specifically, there is a $W\in SL_n(\mathbb{Z})$ and a sequence of 2×2 matrices $S(m_i)=\begin{pmatrix}0 & -m_i\\ m_i & 0\end{pmatrix}$, $i=1,\dots,N$, with $m_i\in\mathbb{Z}$ for each $i=1,\dots,N$, where $rankH=2N$, such that

$$W\cdot H\cdot W^t=\mathrm{Diag}(S(m_1),\dots,S(m_N),0,\dots,0). \tag{3.2}$$

Any matrix of the form of the right-hand side in (3.2) will be called a canonical form of H.

Thus, a canonical form of H reduces the algebra to the tensor product of twisted Laurent polynomial algebras in two variables with commutation relation $xy=q^r yx$. By Theorem 3.2, it follows in particular that the degree of a twisted Laurent polynomial algebra in two variables is equal to $m/(m,r)$, where (m,r) is the greatest common divisor of m and r.

4. The center and the degree of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$

In this section, we compute the center and the degree of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ by using the associated quasipolynomial algebra $\overline{\mathcal{O}_q(sp\mathbb{C}^{2n})}$.

The associated quasipolynomial algebra $\overline{\mathcal{O}_q(sp\mathbb{C}^{2n})}$ is an associative algebra generated by $\bar{x}_1,\dots,\bar{x}_{2n}$ satisfying the following relations:

$$\begin{aligned}&\bar{x}_j\bar{x}_i=q\bar{x}_i\bar{x}_j,\ \ i<j,\ i'\neq j\\ &\bar{x}_{i'}\bar{x}_i=q^2\bar{x}_i\bar{x}_{i'}\quad i<i',\end{aligned} \tag{4.1}$$

where $i'=2n-i+1$ for $1\le i\le 2n$.

Theorem 4.1. *Let q be a primitive mth root of unity with m an odd positive integer. Then the degree of the algebra $\overline{\mathcal{O}_q(sp\mathbb{C}^{2n})}$ is m^n and the center of $\overline{\mathcal{O}_q(sp\mathbb{C}^{2n})}$ is generated by $\bar{x}_i^m$ for all i.*

Proof. Clearly, $\bar{x}_i^m$ is central for all i and so the degree of the algebra $\overline{\mathcal{O}_q(sp\mathbb{C}^{2n})}\le m^n$. It is well known that the degree of the algebra is equal to the maximal dimension of the irreducible representations of the algebra. Hence, if we can construct an irreducible representation of the algebra $\overline{\mathcal{O}_q(sp\mathbb{C}^{2n})}$ of dimension m^n, everything will be done. Now, our task is to construct an irreducible representation on the space $(\mathbb{C}^m)^n$.

Let $\sigma, D \in End(\mathbb{C}^m)$ such that with respect to the standard basis $v_0, v_1, \ldots, v_{m-1}$

$$\sigma(v_i) = v_{i+1},\ D(v_i) = q^i v_i \quad \text{for all } i. \tag{4.2}$$

We denote by σ_i and similarly D_i the operators $1 \otimes 1 \otimes \cdots \otimes \sigma \otimes 1 \cdots \otimes 1$ and $1 \otimes 1 \otimes \cdots \otimes D \otimes 1 \cdots \otimes 1$ on $(\mathbb{C}^m)^n$ with σ and D in the ith positions for $i = 1, 2, \ldots, n$.

Let

$$\begin{aligned} \bar{x}_i &= D_i \sigma_i \sigma_{i+1} \cdots \sigma_n \quad \text{for } i = 1, 2, \ldots, n; \\ \bar{x}_{i'} &= D_i \sigma_i^{-1} \sigma_{i+1}^{-1} \cdots \sigma_n^{-1} \quad \text{for } i = 1, 2, \ldots, n. \end{aligned} \tag{4.3}$$

By checking the defining relations of the algebra $\overline{\mathcal{O}_q(sp\mathbb{C}^{2n})}$, we see that the above formula gives a representation of the algebra $\overline{\mathcal{O}_q(sp\mathbb{C}^{2n})}$ on the space $(\mathbb{C}^m)^n$. Note that $\bar{x}_i \bar{x}_{i'} = D_i^2$, $D_i^m = 1$, we get

$$D_i = (\bar{x}_i \bar{x}_{i'})^{(m+1/2)}. \tag{4.4}$$

By (4.3), all of the σ_i's can be generated by $\bar{x}_i$'s. Therefore, the above representation is irreducible. □

By the defining relation of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$, we get

Lemma 4.2. *For any positive integer s,*

$$x_i x_{i'}^s = q^{-2s} x_{i'}^s x_i + (q^{-2s} - 1) q^{-1} \Omega_{i-1} x_{i'}^{s-1} \quad \textit{for } i = 1, 2, \ldots, n, \tag{4.5}$$

$$x_{i'} x_i^s = q^{2s} x_i^s x_{i'} + (q^{2s} - 1) q \Omega_{i-1} x_i^{s-1} \quad \textit{for } i = 1, 2, \ldots, n, \tag{4.6}$$

$$x_{i'} x_i^t = x_i^t x_{i'} + (q^{2t} - 1) x_i^{t-1} \Omega_i, \tag{4.7}$$

$$x_i x_{i'}^t = x_{i'}^t x_i + q^2 (q^{-2t} - 1) x_{i'}^{t-1} \Omega_i. \tag{4.8}$$

Hence x_i^m and $x_{i'}^m$ are central elements provided that q is a mth root of unity.

Theorem 4.3. *Let m be an odd positive integer and let q be a primitive mth root of unity. Then the degree of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ is m^n. The center of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ is generated by x_i^m for all $i = 1, 2, \ldots, 2n$.*

Proof. By Lemma 2.4 and Theorems 3.1 and Theorem 4.1, the degree of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ is m^n. By Lemma 4.2, the center of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ is generated by x_i^m for all $i = 1, 2, \ldots, 2n$. □

The following lemmas are mainly due to Jakobsen which will be needed in the next section.

Let $p_i = (1 - q^{-2i})$. For $i, j \in \mathbb{N}$ set $S_{i,j} = \{a \in \mathbb{N} | 1 \le a \le j, \text{ and } a \ne i\}$. Let

$$F_{i,j}(n) = \frac{p_i^{n-1}}{\prod_{a \in S_{i,j}} (p_i - p_a)}. \tag{4.9}$$

Lemma 4.4. *Let*

$$(x_i x_{i'})^n = \sum_{t=0}^{n} a_t(n) x_i^t x_{i'}^t \Omega_i^{n-t}. \tag{4.10}$$

Then

$$a_t(n) = \sum_{b=1}^{t} F_{b,t}(n). \tag{4.11}$$

Moreover, since the right-hand side makes sense for all n, we can define the left-hand side by this formula. Doing this, we get that $a_t(s) = 0$ for $s = 1, \ldots, t-1$.

Proof. We have

$$\begin{aligned}(x_i x_{i'})^{n+1} &= x_i x_{i'} \sum_{t=0}^{n} a_t(n) x_i^t (x_{i'})^t \Omega_i^{n-t} \\ &= \sum_{t=0}^{n} a_t(n) x_i x_{i'} x_i^t (x_{i'})^t \Omega_i^{n-t} \\ &= \sum_{t=0}^{n} a_t(n) x_i (x_i^t x_{i'} + (q^{2t} - 1) x_i^{t-1} \Omega_i) x_{i'}^t \Omega_i^{n-t} \\ &= \sum_{t=0}^{n} (a_t(n) x_i^{t+1} x_{i'}^{t+1} \Omega_i^{n-t} + (1 - q^{-2t}) a_t(n) x_i^t x_{i'}^t \Omega_i^{n-t+1}).\end{aligned} \tag{4.12}$$

Hence, if we set $a_{-1}(n) = a_{n+1}(n) = 0$, we get

$$\forall t = 0, \ldots, n+1: \quad a_t(n+1) = (1 - q^{-2t}) a_t(n) + a_{t-1}(n). \tag{4.13}$$

In particular,

$$\forall n{:} a_n(n) = 1 \quad \text{and} \quad \forall n{:} a_0(n) = 0. \tag{4.14}$$

It is straightforward to check that (4.13) is satisfied by (4.11). As for (4.14) as well as the remaining assertion, observe that

$$\sum_{b=1}^{t} \frac{p_b^{s-1}}{\prod_{a \in S_{i,j}} (p_b - p_a)} \tag{4.15}$$

is the (t,s)th entry of the matrix product $A^{-1} \cdot A$ where A is the (Vandermonde) matrix

$$A = \begin{pmatrix} 1 & p_1 & p_1^2 & \cdots & p_1^{t-1} \\ 1 & p_2 & p_2^2 & \cdots & p_2^{t-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & p_t & p_t^2 & \cdots & p_t^{t-1} \end{pmatrix}. \quad \square \tag{4.16}$$

Lemma 4.5.

$$\Omega_i^m = \sum_{k=1}^{i} x_k^m x_{k'}^m. \tag{4.17}$$

Proof. Observe that

$$\Omega_i^m = (x_i x_{i'} + \Omega_{i+1})^m, \tag{4.18}$$

since $x_i x_{i'}$ and Ω_{i+1} commute we have

$$\Omega_i^m = \sum_{n=0}^{m} \binom{m}{n} (x_i x_{i'})^n \Omega_{i+1}^{m-n}. \tag{4.19}$$

Therefore

$$\Omega_i^m = \sum_{n=0}^{m} \binom{m}{n} \sum_{t=0}^{n} a_t(n) x_i^t x_{i'}^t \Omega_{i+1}^{m-t}. \tag{4.20}$$

The coefficient of $x_i^t x_{i'}^t \Omega_{i+1}^{m-t}$ is

$$\sum_{n=t}^{m} \binom{m}{n} a_t(n). \tag{4.21}$$

By Lemma 4.4, this may be written

$$\sum_{n=1}^{m} \binom{m}{n} \sum_{b=1}^{t} \frac{p_b^{n-1}}{\prod_{a\in S_{i,j}}(p_b - p_a)} = \sum_{b=1}^{t} \sum_{n=1}^{m} \binom{m}{n} \frac{p_b^{n-1}}{\prod_{a\in S_{i,j}}(p_b - p_a)}. \tag{4.22}$$

The coefficient is clearly 1 if $t = m$. For all other values of t it is zero since then $p_b \neq 0$ and thus

$$\sum_{n=1}^{m} \binom{m}{n} p_b^{n-1} = \frac{1}{p_b}((p_b + 1)^m - 1) = 0. \qquad \square \tag{4.23}$$

5. The irreducible representations of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$

In this section, the De Concini–Kac–Procesi Conjecture is proved to be true for the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$.

Let A be an associative algebra with generators $x_1, x_2, \ldots, x_n$ and the defining relations:

$$x_i x_j = q^{h_{ij}} x_j x_i + p_{ij}, \tag{5.1}$$

where $i > j$, $p_{ij} \in \mathbb{C}[x_1, \ldots, x_{i-1}]$, (h_{ij}) is an anti-symmetric integral matrix. Let q be a primitive mth root of unity. Assume that x_i^m is central for all i. Let $\eta_1, \eta_2, \ldots, \eta_n$ be the standard coordinate functions of $\mathbb{C}^n$. There is a Poisson structure on $\mathbb{C}^n$ induced by the defining relations of the algebra A:

$$\{\eta_i, \eta_j\} = \left.\frac{[x_i, x_j]}{q-1}\right|_{x\to\eta, q=1}. \tag{5.2}$$

The maximal integral symplectic submanifolds of $\mathbb{C}^n$ are called the symplectic leaves of $\mathbb{C}^n$.

Let π be an irreducible representation of the algebra A. Then there exists $p = (p_1, p_2, \ldots, p_n) \in \mathbb{C}^n$ such that $x_i^m = p_i$ on π. Assume that $\mathcal{O}_\pi$ is a symplectic leaf contains the point p.

The following conjecture is posed by De Concini et al. [1].

DKP Conjecture. $\mathrm{d}im\,\pi = m^{\frac{1}{2}\mathrm{d}im\mathcal{O}_\pi}$, if m is prime to all of the elementary divisors of the integral matrix H.

At first, we need some remarks from [1]. Assume that we have a manifold M and a vector bundle V of algebras with 1 (i.e. 1 and the multiplication map are smooth sections). We identify the functions on M with the sections of V which are multiples of 1. Let D be a derivation of V, i.e. a derivation of the algebra of sections which maps the algebra of functions on M into itself. Let X be the corresponding vector field on M.

The following two propositions were proved in [1].

Proposition 5.1. *For each point $p \in M$ there exists a neighborhood U_p and a map ϕ_t defined for $|t|$ sufficiently small on $V|U_p$ which is a morphism of vector bundles covering the germ of the* 1*-parameter group generated by X is also an isomorphism of the algebras.*

Now, suppose M is a Poisson manifold. Assume furthermore that the Poisson structure lifts to V, i.e. each local function f induces a derivation on sections extending the given Poisson bracket. This means that we have a lift of the Hamiltonian vector field.

Proposition 5.2. *Under the above hypotheses, the fibres of V over the points of a given symplectic leaf M are all isomorphic as algebras.*

The Poisson structure of $\mathbb{C}^{2n}$ induced by the defining relations of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ is defined by the following formulas:

$$\begin{aligned} &\{a_j, a_i\} = a_i a_j \quad \text{for } i < j,\ j \neq 2n - i + 1, \\ &\{a_{i'}, a_i\} = 2 \sum_{1 \le k \le i} a_k a_{k'} \quad \text{for } i = 1, 2, \ldots, n, \end{aligned} \tag{5.3}$$

for any point $p = (a_1, a_2, \ldots, a_{2n}) \in \mathbb{C}^{2n}$.

On a symplectic leaf, the fibres of the algebra are isomorphic. Hence, our strategy to study the representation of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ on a symplectic leaf is to choose a good point p and construct a quasipolynomial algebra such that any irreducible representation of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ at the point p is still irreducible viewed as a representation of the quasipolynomial algebra.

Lemma 5.3. *On each symplectic leaf $\mathcal{O}$, there is a point*

$$p_{\mathcal{O}} = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_{2n}) \tag{5.4}$$

such that one of $a_i, a_{i'}$ is nonzero or both $a_i = a_{i'} = 0$ and $\sum_{1 \le k \le i} a_k a_{k'} = 0$.

Proof. Let $p_{\mathcal{O}} = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_{2n}) \in \mathcal{O}$ such that the number of the nonzero coordinates in $p_{\mathcal{O}}$ is maximal in $\mathcal{O}$. If $a_i = a_{i'} = 0$ but $\sum_{1 \le k \le i} a_k a_{k'} \neq 0$. We apply the Hamiltonian vector field $Z_{x_{i'}}$ induced by $x_{i'}$. Let

$$\alpha(t) = (a_1(t), \ldots, a_{2n}(t)) \tag{5.5}$$

be an integral curve of the vector field $Z_{x_{i'}}$ with the initial point $p_{\mathcal{O}}$. Then for any $j \neq i'$ $a_j(t)$ is of the form

$$a_j(t) = a_j \mathrm{e}^{a_{i'} t} \quad \text{or} \quad a_j \mathrm{e}^{-a_{i'} t} \tag{5.6}$$

and so the $a_j(t)$ is nonzero if $a_j \neq 0$. Note that $a_i(t)' = \sum_{1 \le k \le i} a_k(t) a_{k'}(t)$ is nonzero when $t = 0$, so there is a t such that $a_i(t) \neq 0$. Hence, by applying the Hamiltonian vector field $Z_{x_{i'}}$, we can flow from the point $p_{\mathcal{O}}$ to another point p' with more nonzero coordinates in the symplectic leaf $\mathcal{O}$. This is a contradiction. $\square$

Let $\mathbb{C}_q[\mathcal{O}, p_{\mathcal{O}}]$ be the subalgebra generated by the following three kinds of elements:
1. x_i if $a_i \neq 0$ for $1 \le i \le n$.
2. x_j if $a_{j'} = 0$ and $a_j \neq 0$ for $n+1 \le j \le 2n$.
3. Ω_k for all $1 \le k \le n$ such that $\omega_k := \sum_{i=1}^{k} a_i a_{i'} \neq 0$.

Clearly, the subalgebra $\mathbb{C}_q[\mathcal{O}, p_{\mathcal{O}}]$ is a quasipolynomial algebra and we call it the quasipolynomial algebra associated to the good point $p_{\mathcal{O}}$.

Note that if we restrict our attention to the good point $p_{\mathcal{O}}$, the generators of the quasipolynomial algebra $\mathbb{C}_q[\mathcal{O}, p_{\mathcal{O}}]$ we chose are all invertible when acting on an irreducible module at the point $p_{\mathcal{O}}$, so any irreducible $\mathcal{O}_q(sp\mathbb{C}^{2n})$-module can be viewed as a module over the Laurent quasipolynomial algebra $L\mathbb{C}_q[\mathcal{O}, p_{\mathcal{O}}]$.

Now, we can prove our main result.

Theorem 5.4. *Let q be a primitive mth root of unity, where m is an odd positive integer. Let V be an irreducible $\mathcal{O}_q(sp\mathbb{C}^{2n})$-module at a point $p = (a_1, a_2, \ldots, a_{2n})$ and let $\mathcal{O}$ be the symplectic leaf contains p. Then*

$$\mathrm{d}im\, V = m^{\frac{1}{2}\mathrm{d}im\, \mathcal{O}}. \tag{5.7}$$

Proof. Since the fibres of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ over the points of a given symplectic leaf are all isomorphic as algebras, one can assume that the point p coincides with the good point $p_{\mathcal{O}}$. Now, V is an irreducible representation of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$ at the point $p_{\mathcal{O}}$. If $a_i = a_{i'} = 0$, then both x_i and $x_{i'}$ are nilpotent, on V. Since Ω_i is nilpotent by (4.17) and the covariance of Ω_i, we see that $\Omega_i = 0$ when acting on the irreducible representation V. By the defining relation of the algebra $\mathcal{O}_q(sp\mathbb{C}^{2n})$, the generators x_i and $x_{i'}$ are also covariant when acting on V and therefore vanish on the irreducible representation V. If one of $a_i, a_{i'}$ is nonzero, we assume that $a_i \neq 0$, without losing generality, then obviously $x_{i'}$ also belongs to the algebra $L\mathbb{C}_q[\mathcal{O}, p_{\mathcal{O}}]$. Hence, as a $L\mathbb{C}_q[\mathcal{O}, p_{\mathcal{O}}]$ module, V is still irreducible. It is well known that the Laurent quasipolynomial algebra $L\mathbb{C}_q[\mathcal{O}, p_{\mathcal{O}}]$ is an Azumaya algebra over its center, so the dimension of the irreducible representation of the algebra $L\mathbb{C}_q[\mathcal{O}, p_{\mathcal{O}}]$ is equal to the degree of this Laurent quasipolynomial algebra.

By the Leibniz rule,

$$\{a_i, \omega_j\} = \left\{ a_i, \sum_{k=1}^{j} a_k a_{k'} \right\} = \sum_{k=1}^{j} \{a_i, a_k\} a_{k'} + \sum_{k=1}^{j} a_k \{a_i, a_{k'}\} \tag{5.8}$$

and Lemma 5.3, we see easily that the matrix $(\{a_s, a_t\})_{2n\times 2n}$ is equivalent to the matrix

$$\begin{pmatrix} (\{a_i, a_j\})_{i,j=1}^{n} & (\{a_i, \omega_j\})_{i,j=1}^{n} \\ (\{\omega_j, a_i\})_{i,j=1}^{n} & 0 \end{pmatrix}. \tag{5.9}$$

By a direct computation, one can see easily that the above matrix is equivalent to the defining matrix of the Laurent quasipolynomial algebra $L\mathbb{C}_q[\mathcal{O}, p_{\mathcal{O}}]$ if we expand the defining matrix by adding certain zero rows and columns to make it into a $2n \times 2n$ matrix. By Theorems 3.1 and 3.2, the degree of the Laurent quasipolynomial algebra $L\mathbb{C}_q[\mathcal{O}, p_{\mathcal{O}}]$ is equal to $m^{\frac{1}{2}\mathrm{d}im\,\mathcal{O}}$ if q is a primitive mth root of unity and m is prime to all of the elementary divisors of the defining matrix of the algebra $L\mathbb{C}_q[\mathcal{O}, p_{\mathcal{O}}]$.

To compute the degree of this Laurent quasipolynomial algebra, we may assume that the generators x_i's of $L\mathbb{C}_q[\mathcal{O}, p_{\mathcal{O}}]$ are all from the set $\{x_1, x_2, \ldots, x_n\}$, since if x_i is nilpotent but $x_{i'}$ is invertible, we can choose $x_{i'}^{-1}$ which has the same relations with the other generators as x_i does. Note that if x_i and x_j are both invertible for $i < j$ and x_k is nilpotent for all $i < k < j$. By the choice of the generators above, we see that x_k is nilpotent implies that $x_{k'}$ is also nilpotent and this also implies that $\Omega_k = 0$. Therefore, $x_k = x_{k'} = 0$. So after the modifications we have just made we may assume that all x_i's are invertible. Hence, what we need to do is to compute the degree of the Laurent quasipolynomial algebra

$$L_q(i_1, i_2, \ldots, i_s; j_1, j_2, \ldots, j_r)$$

generated by

$$x_{i_1}, x_{i_2}, \ldots, x_{i_s} \quad \text{and} \quad \Omega_{j_1}, \ldots, \Omega_{j_r}$$

where $j_1, \ldots, j_r \in \{i_1, i_2, \ldots, i_s\}$ and $j_1 < j_2 < \cdots < j_r$. For convenience, we have only to compute the degree of the Laurent quasipolynomial algebra $L(1, 2, \ldots, n; j_1, j_2, \ldots, j_r)$ generated by $x_1, x_2, \ldots, x_n, \Omega_{j_1}, \ldots, \Omega_{j_r}$ and we denote the algebra by

$$L_q(1, 2, \ldots, n; j_1, j_2, \ldots, j_r).$$

If $j_r < n$, we consider the Laurent subalgebra generated by

$$x_1, \ldots, x_{j_r}, \Omega_{j_1}, \ldots, \Omega_{j_r} \quad \text{and} \quad x_{j_r+1}^2 \Omega_{j_r}^{-1}, \ldots, x_n^2 \Omega_{j_r}^{-1}.$$

Note that the degree of this subalgebra is equal to the degree of the algebra

$$L_q(1,2,\dots,n;j_1,j_2,\dots,j_r)$$

if q is a primitive mth root of unity and m is any odd positive integer. The algebra above is a direct product of

$$L_q(1,2,\dots,j_r;j_1,j_2,\dots,j_r)$$

and the algebra $L_{q^2}(j_r+1,\dots,n)$ generated by

$$x_{j_r+1}^2\Omega_{j_r}^{-1},\dots,x_n^2\Omega_{j_r}^{-1}.$$

The degree of the algebra $L_{q^2}(j_r+1,\dots,n)$ is $m^{u(n-j_r)/2}$ if q is a primitive mth root of unity and m is any odd positive integer.

To compute the degree of the algebra $L_q(1,2,\dots,j_r;j_1,j_2,\dots,j_r)$, we consider the Laurent subalgebra generated by

$$x_1^2x_{j_r}^{-2}\Omega_{j_r},\dots,x_{j_r-1}^2x_{j_r}^{-2}\Omega_{j_r},\Omega_{j_1},\Omega_{j_2},\dots,\Omega_{j_{r-1}} \quad \text{and} \quad x_{j_r},\Omega_{j_r}.$$

Clearly, the above algebra is a direct product of $L_{q^4}(1,2,\dots,j_r-1;j_1,\dots,j_{r-1})$ and the algebra generated by x_{j_r},Ω_{j_r} if q is a primitive mth root of unity and m is any odd positive integer. It is easy to see that any irredducible $L_q(1,2,\dots,j_r;j_1,j_2,\dots,j_r)$ module is also an irreducible

$$L_{q^4}(1,2,\dots,j_r-1;j_1,\dots,j_{r-1})\times\langle x_{j_r},\Omega_{j_r}\rangle$$

module. The degree of the algebra $L_q(1,2,\dots,j_r;j_1,j_2,\dots,j_r)$ is equal to

$$m(\text{the degree of the algebra } L_{q^4}(1,2,\dots,j_r-1;j_1,\dots,j_{r-1})). \tag{5.10}$$

By induction on r, we see that the rank of defining matrix of the algebra

$$L_q(1,2,\dots,n;j_1,j_2,\dots,j_r)$$

is

$$2r+2\sum_{k=2}^{r}\left[\frac{j_k-j_{k-1}}{2}\right]+2\left[\frac{j_1-1}{2}\right],$$

if q is a primitive mth root of unity m is any odd positive integer. This is also implies that the nonzero entries of the canonical form of the defining matrix of the algebra

$$L_q(i_1,i_2,\dots,i_s;j_1,j_2,\dots,j_r)$$

are all of the form 2^t or -2^t, so the elementary divisors of this matrix are all of the form 2^t for some $t=0,1,\dots$. This completes the proof. □

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